Hierarchy of general invariants for bivariate LPDOs.

E. Kartashova

RISC, J.Kepler University, Linz, Austria

e-mail: lena@risc.uni-linz.ac.at

Abstract

We study invariants under gauge transformations of linear partial differential operators on two variables. Using results of BK-factorization, we construct hierarchy of general invariants for operators of an arbitrary order. Properties of general invariants are studied and some examples are presented. We also show that classical Laplace invariants correspond to some particular cases of general invariants.

Keywords: linear partial differential operator, BK-factorization, general invariant, hierarchy of invariants

1 Introduction

Laplace invariants $\hat{a} = c - ab - a_x$ and $\hat{b} = c - ab - b_y$, introduced for bivariate hyperbolic operator of the second order

$$\partial_x \partial_y + a\partial_x + b\partial_y + c, \tag{1}$$

are quite well-known and have been studied by many researchers (see, for instance, [7] and bibl. herein). Their importance is due to the classical theorem:

Theorem 1 Two operators of the form are equivalent under gauge transformations **iff** when their Laplace invariants coincide pairwise.

These invariants are used extensively in the theory of integrability ([8], [6], [2], [11], [9],[10] and others). Obviously, Laplace invariants can be regarded as factorization's "remainders" for the initial operator (1):

$$\partial_x \partial_y + a\partial_x + b\partial_y + c = \begin{cases} (\partial_x + b)(\partial_y + a) - ab - a_x + c \\ (\partial_y + a)(\partial_x + b) - ab - b_y + c \end{cases} \tag{2}$$

and this presentation is a first step of Darboux-Laplace transformations used for construction of solution to non-factorizable linear partial differential equations in two independent variables.

To generalize construction of invariants to the case of an arbitrary order operator, a factorization algorithm for such an operator is needed. Recently two factorization

algorithms for arbitrary order bivariate LPDOs have been published. In [3], the factorization algorithm called *Hensel descent* is presented, which is a modification of well-known Hensel lifting algorithm (see, for instance, [4]) and which allows to factorize operators from the ring $D = \mathbb{Q}(x, y)[\partial_x, \partial_y]$. In [1], *absolute* factorization is presented for operators whose coefficients are arbitrary smooth functions of two variables. The idea to connect the results produced by this method (called now BK-factorization) and Darboux-Laplace transformations has been briefly discussed in [5]. In present paper we construct general invariants using BK-factorization procedure which can be described as follows. Factorization problem is regarded in the form

$$A_n = \sum_{j+k\le n} a_{jk}\partial_x^j\partial_y^k = (p_1\partial_x + p_2\partial_y + p_3)\Big(\sum_{j+k<n} p_{jk}\partial_x^k\partial_y^j\Big) \tag{3}$$

with assumption that $a_{n0} \neq 0$ (locally) and can be taken as $a_{n0} = 1$ without loss of generality. It is shown that for any simple root ω of characteristic polynomial

$$A(\omega) = \sum_{k=0}^{n} a_{n-k,k}\omega^{n-k},$$

with $n \ge 2$, there exists unique factorization of the form (3):

$$A = LB + C, \qquad L = \partial_x - \omega\partial_y + p, \qquad C = \sum_{j=1}^{n-2} c_j\partial_y^j. \tag{4}$$

and the conditions of factorization are written out explicitly. In Section 2 we use these results for construction of general invariants of an arbitrary LPDO and study some of their properties while in Section 3 the whole hierarchy of general invariants is described. In Section 4 a few instructive examples are given. In particular, it is shown that both Laplace invariants can be constructed as a simple particular case of general invariants. In Section 5 brief discussion of the results obtained is presented.

2 General invariants and semi-invariants

The conditions of factorization produced by BK-factorization have following form:

for second order operator

$$A_2 = \sum_{j+k\le 2} a_{jk}\partial_x^j\partial_y^k = (p_1\partial_x + p_2\partial_y + p_3)(p_4\partial_x + p_5\partial_y + p_6)$$

the only factorization condition is

$$a_{00} = \mathcal{L}(p_6) + p_3p_6, \tag{5}$$

for third order operator

$$A_3 = \sum_{j+k\le 3} a_{jk}\partial_x^j\partial_y^k = (p_1\partial_x + p_2\partial_y + p_3)(p_4\partial_x^2 + p_5\partial_x\partial_y + p_6\partial_y^2 + p_7\partial_x + p_8\partial_y + p_9)$$

there are two factorization conditions:

$$\begin{cases} a_{01} = & \mathcal{L}(p_8) + p_3p_8 + p_2p_9 \\ a_{00} = & \mathcal{L}(p_9) + p_3p_9 \end{cases} \tag{6}$$

and so on for arbitrary n. Now we can give explicit representation of a LPDO whose characteristic polynomial has at least one simple root, in the form of factorization with remainders:

$$A_2 = (p_1\partial_x + p_2\partial_y + p_3)(p_4\partial_x + p_5\partial_y + p_6) - l_2$$

with remainder $l_2 = a_{00} - L(p_6) + p_3 p_6$;

$$A_3 = (p_1\partial_x + p_2\partial_y + p_3)(p_4\partial_x^2 + p_5\partial_x\partial_y + p_6\partial_y^2 + p_7\partial_x + p_8\partial_y + p_9) - l_3\partial_y - l_{31},$$

with two remainders $l_3 = a_{01} - (p_1\partial_x + p_2\partial_y + p_3)p_8 - p_2 p_9$, and $l_{31} = a_{00} - (p_1\partial_x + p_2\partial_y + p_3)p_9$; and so on for arbitrary order of LPDO.

Definition The operators A, $\tilde{A}$ are called equivalent if there is a gauge transformation that takes one to the other:

$$\tilde{A}g = e^{-\varphi}A(e^{\varphi}g) \equiv A_{\varphi}g.$$

Note that

$$(\partial_x)_{\varphi} = \partial_x + \varphi_x, \quad (\partial_y)_{\varphi} = \partial_y + \varphi_y$$

and a gauge transformation is an algebra automorphism. Therefore A and A_{φ} have the same characteristic polynomial and the factorization (4) carries over:

$$A_{\varphi} = L_{\varphi}B_{\varphi} + C_{\varphi}.$$

It follows that both the characteristic polynomial and the leading nonzero coefficient of the remainder term are invariants. Number of remainders varies for operators of different orders, and also their properties are different. In order to demonstrate it, let us formulate the following

Lemma 1 For an operator of order 2, its remainder l_2 is its **invariant** under the equivalence transformation, i.e.

$$\tilde{l}_2 = l_2.$$

For an operator of order 3, its remainder l_3 is its **invariant**, i.e.

$$\tilde{l}_3 = l_3,$$

while remainder l_{31} changes its form as follows:

$$\tilde{l}_{31} = l_{31} + l_3\varphi_y.$$

▶ Introducing notations $A_{2a} = A_{2p} - l_2$, $A_{3a} = A_{3p} - l_3\partial_y - l_{31}$, $f = e^{\varphi}$, we get

$$\tilde{A}_{2a} = f^{-1}A_{2a} \circ f = f^{-1}(A_{2p} - l_2) \circ f = \tilde{A}_{2p} - f^{-1}(l_2) \circ f = \tilde{A}_{2p} - l_2.$$

and

$$\tilde{A}_{3a} = f^{-1}A_{3a} \circ f = f^{-1}(A_{3p} - l_3\partial_y - l_{31}) \circ f = \tilde{A}_{3p} - f^{-1}(l_{31} + l_3\partial_y) \circ f = \tilde{A}_{3p} - l_3\partial_y - l_3\varphi_y - l_{31}. \blacksquare$$

Corollary 1: If $l_3 = 0$, then l_{31} becomes invariant.

That is the reason why we call l_{31} further **semi-invariant**.

Corollary 2: If $l_3 \neq 0$, it is always possible to choose some function f : $\tilde{l}_{31} = l_3\varphi_y + l_{31} = 0$.

Note that for second order operator, if its invariant $l_2 = 0$ then operator is factorizable while for third order operator two its invariants have to be equal to zero, $l_3 = l_{31} = 0$. On the other hand, if operator of third order is not factorizable we can always regard it as an operator with only one non-zero invariant. Of course, all this is true for each simple root of characteristic polynomial, so that one expression, say, for l_3, will generate three invariants in case of three simple roots of corresponding polynomial.

3 Hierarchy of invariants

As it was shown above, every general invariant is a function of a simple root ω of the characteristic polynomial and each simple root provides one invariant. It means that for operator of order n we can get no more than n different invariants. Recollecting that BK-factorization in this case gives us one first order operator and one operator of order $n-1$, let us put now following question: are general invariants of operator of order $n-1$ also invariants of corresponding operator of order n?

Let regard, for instance, operator of order 3

$$A_{3a} = A_1A_{2a} - l_3\partial_y - l_{31} = A_1(A_{2p} - l_2) - l_3\partial_y - l_{31} = A_1A_{2p} - l_2A_1 - l_3\partial_y - l_{31}$$

and

$$\tilde{A}_{3a} = A_1A_{2p} - l_2A_1 - l_3\partial_y - l_{31} = \tilde{A}_1\tilde{A}_{2p} - l_2\tilde{A}_1 - l_3\partial_y - \tilde{l}_{31},$$

i.e. l_2 is also invariant of operator A_{3a}. Let us notice that general invariant $l_3 = l_3(\omega^{(3)})$ is a function of a simple root $\omega^{(3)}$ of the polynomial

$$\mathcal{P}_3(z) = a_{30}z^3 + a_{21}z^2 + a_{12}z + a_{03}$$

while general invariant $l_2 = l_2(\omega^{(2)})$ is a function of a simple root $\omega^{(2)}$ of the polynomial

$$\mathcal{R}_2(z) = p_4z^2 + p_5z + p_6$$

with $p_4,\ p_5,\ p_6$ given explicitly for $\omega = \omega^{(3)}$. In case of all simple roots of both polynomials $\mathcal{P}_3(z)$ and $\mathcal{R}_2(z)$, one will get maximal number of invariants, namely 6 general invariants. Repeating the procedure for an operator of order n, we get maximally $n!$ general invariants. In this way for operator of arbitrary order n we can construct the hierarchy of its general invariants

$$l_n, l_{n-1}, ..., l_2$$

and their explicit form is given by BK-factorization. As to semi-invariants, notice that an operator of arbitrary order n can always be rewritten in the form of factorization with remainder of the form

$$l_n\partial_x^k + l_{n,1}\partial_x^{k-1} + ... + l_{n,k-1}, \quad k < n$$

and exact expressions for all l_i are provided by BK-factorization procedure. The same reasoning as above will show immediately that l_n is always general invariant, and each $l_{n,k-i_0}$ is i_0-th semi-invariant, i.e. it becomes invariant in case if $l_{n,k-i} = 0,\ \forall i < i_0$.

Example 1 Let us regard a third order hyperbolic operator in the form

$$C = a_{30}\partial_x^3 + a_{21}\partial_x^2\partial_y + a_{12}\partial_x\partial_y^2 + a_{03}\partial y^3 + \text{terms of lower order} \qquad (7)$$

with constant high order coefficients, i.e. $a_{ij} = \text{const} \ \forall \ i + j = 3$ and all roots of characteristic polynomial

$$a_{30}\omega^3 + a_{21}\omega^2 + a_{12}\omega + a_{03} = \mathcal{P}_3(\omega)$$

are simple and real. Then we can construct three simple independent general invariants in following way. Notice first that in this case high terms of (7) can be written in the form

$$(\alpha_1\partial_x + \beta_1\partial_y)(\alpha_2\partial_x + \beta_2\partial_y)(\alpha_3\partial_x + \beta_3\partial_y)$$

for all non-proportional α_j, β_i and after appropriate change of variables this expression can easily be reduced to $\partial_x\partial_y(\partial_x + \partial_y)$. Let us introduce notations $\partial_1 = \partial_x, \quad \partial_2 = \partial_y, \quad \partial_3 = \partial_1 + \partial_2 = \partial_t$, then all terms of the third and second order can be written out as

$$C_{ijk} = (\partial_i + a_i)(\partial_j + a_j)(\partial_k + a_k) = \partial_i\partial_j\partial_k + a_k\partial_i\partial_j + a_j\partial_i\partial_k + a_i\partial_j\partial_k +$$
$$+(\partial_j + a_j)(a_k)\partial_i + (\partial_i + a_i)(a_k)\partial_j + (\partial_i + a_i)(a_j)\partial_k + (\partial_i + a_i)(\partial_j + a_j)(a_k)$$

with

$$a_{20} = a_2, \ a_{02} = a_1, \ a_{11} = a_1 + a_2 + a_3$$

and $c_{ijk} = C - C_{ijk}$ is an operator of the first order which can be written out explicitly. As it was shown above, coefficients of c_{ijk} in front of first derivatives are (semi)invariants and therefore, any linear combination of invariants is an invariant itself. Direct calculation gives us three simplest general invariants of the initial operator C:

$$l_{21} = a_{2,x} - a_{1,y}, \ l_{32} = a_{3,y} - a_{2,t}, \ l_{31} = a_{3,x} - a_{1,t}.$$

Lemma 2 General invariants $l_{21}, \ l_{32}, \ l_{31}$ are all equal to zero **iff** operator C is equivalent to an operator

$$L = \partial_1\partial_2\partial_3 + b_1\partial_1 + b_2\partial_2 + c, \qquad (8)$$

i.e. $\exists$ function $f: \quad f^{-1}C \circ f = L$.

► Obviously

$$f^{-1}(\partial_x\partial_y\partial_t) \circ f = (\partial_x + (\log f)_x)(\partial_y + (\log f)_y)(\partial_t + (\log f)_t)$$

for any smooth function f. Notice that it is the form of an operator C_{ijk} and introduce a function f such that

$$a_1 = (\log f)_x, \quad a_2 = (\log f)_y, \quad a_3 = (\log f)_t.$$

This system of equations on f is over-determined and it has solution f_0 **iff**

$$a_{2,x} - a_{1,y} = 0, \ a_{3,y} - a_{2,t} = 0, \ a_{3,x} - a_{1,t} = 0, \qquad (9)$$

i.e. $l_{21} = l_{32} = l_{31} = 0$.

◄ Indeed, if C is equivalent to (8), then $a_{20} = a_{02} = a_{11} = 0$ and obviously $l_{21} = l_{32} = l_{31} = 0$. ▮

Note that the statement of the Lemma 2 is weaker than that of Theorem 1.

Example 2 First two equations of (9) determine solution uniquely, up to a constant, and therefore determine

$$\partial_3 f = (\partial_1 + \partial_2) f = a_1 + a_2.$$

However the compatibility conditions are satisfied if $\partial_2 a_1 = \partial_1 a_2$ and $a_3 = a_1 + a_2 + c$, c constant. In general, the compatibility conditions only tell that two equations of (9) are satisfied, but the third may be off by a constant. For instance, two constant coefficient operators

$$\partial_x \partial_y (\partial_x + \partial_y), \qquad \partial_x \partial_y (\partial_x + \partial_y + 1)$$

have the same general invariants (all zero) but are not equivalent.

4 Further examples

Example 3 Let us regard hyperbolic operator in the form

$$\partial_{xx} - \partial_{yy} + a_{10}\partial_x + a_{01}\partial_y + a_{00}, \tag{10}$$

i.e. $a_{20} = 1, a_{11} = 0, a_{02} = -1$ and $\omega = \pm 1,\ \mathcal{L} = \partial_x - \omega \partial_y$. Then l_2 takes form

$$l_2 = a_{00} - \mathcal{L}(\frac{\omega a_{10} - a_{01}}{2\omega}) - \frac{\omega a_{10} - a_{01}}{2\omega} \frac{\omega a_{10} + a_{01}}{2\omega}$$

which yields, for instance for the root $\omega = 1$, to

$$l_2 = a_{00} - \mathcal{L}(\frac{a_{10} - a_{01}}{2}) - \frac{a_{10}^2 - a_{01}^2}{4} = a_{00} - (\partial_x - \partial_y)(\frac{a_{10} - a_{01}}{2}) - \frac{a_{10}^2 - a_{01}^2}{4}$$

and after obvious change of variables in (10) we get finally first Laplace invariant $\hat{a}$

$$l_2 = c - \partial_{\tilde{x}} a - ab = \hat{a},$$

where

$$a = \frac{a_{10} - a_{01}}{2}, \quad b = \frac{a_{10} + a_{01}}{2}, \quad c = a_{00}.$$

Choice of the second root, $\omega = -1$, gives us the second Laplace invariant $\hat{b}$, i.e. Laplace invariants are particular cases of the general invariant so that each Laplace invariant corresponds to a special choice of ω.

Example 4 Let us proceed analogously with an elliptic operator

$$\partial_{xx} + \partial_{yy} + a_{10}\partial_x + a_{01}\partial_y + a_{00}, \tag{11}$$

then $\omega = \pm i,\ \mathcal{L} = \partial_x - \omega \partial_y$ and

$$l_2 = a_{00} + (\partial_x \mp i\partial_y)(\frac{\pm a_{10} + a_{01} i}{2}) + i\frac{a_{10}^2 + a_{01}^2}{4}$$

where choice of upper signs corresponds to the choice of the root $\omega = i$ and choice of lower signs corresponds to $\omega = -i$.

Example 5 Now let us regard a third order operator

$$B = \partial_x^2\partial_y + \partial_x\partial_y^2 + a_{11}\partial_x\partial_y + a_{10}\partial_x + a_{01}\partial_y + a_{00}, \tag{12}$$

with $a_{30} = a_{03} = a_{20} = a_{02} = 0$, $a_{21} = a_{12} = 1$ and all other coefficients are functions of x, y. Then (semi)invariants

$$l_3 = \partial_x a_{11} - a_{01} \quad \text{and} \quad l_{31} = \partial_x a_{10} - a_{00}$$

have very simple forms and gives us immediately a lot of information about the properties of operators of the form (12), for instance, these operators are factorizable, i.e. has zero invariants $l_3 = l_{31} = 0$, **iff**

$$a_{11} = \int a_{01}dx + f_1(y), \quad a_{10} = \int a_{00}dx + f_2(y)$$

with two arbitrary functions on y, $f_1(y)$ and $f_2(y)$.

5 Summary

Now, that construction of Laplace invariants has been generalized to the case of arbitrary order and arbitrary type of LPDO, the next obvious step would be to generalize of Darboux-Laplace transformations to this class class of operators, to construct generalized Todda lattice, etc. It would require a deeper study of intrinsic properties of general invariants. For instance, beginning with operator of order 4, their maximal number is bigger then number of coefficients of a given operator,

$$\frac{(n+1)(n+2)}{2} < n! \quad \forall n > 3.$$

It means that general invariants are dependent on each other and it will be a challenging task to extract the subset of independent general invariants, i.e. basis in the finite space of general invariants. Another important step would be to extract a subset providing full analog of Theorem 1 for arbitrary order operators.

Already in the case of three variables, the factorization problem of a corresponding operator and also constructing of its invariants becomes more complicated, even for constant coefficients. The reason of it is that in bivariate case we needed just to factorize leading term polynomial which is always possible over $\mathbb{C}$. It is not the case for more then 2 independent variables where a counter-example is easily to find:

Example 6 A polynomial

$$x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - xz - zy)$$

is factorizable but it is easy to prove that $x^3 + y^3 + z^3$ is not divisible by a linear polynomial

$$\alpha x + \beta y + \gamma z + \delta$$

for any complex coefficients α, β, γ, δ. Thus, some non-trivial conditions are to be found for factorization of polynomials in more then two variables.

Acknowledgements

Author acknowledges support of the Austrian Science Foundation (FWF) under projects SFB F013/F1304. Author is grateful to Prof. A. Shabat for useful discussions during preparing of this paper and also to Prof. S. Tsarev who noticed in particular that hierarchy of invariants constructed above can be regarded as analog of classical Todda lattice. Author is also very much obliged to an anonymous reviewer and Prof. R. Beals for their valuable help during preparing of the last version of this paper.

References

[1] R. Beals, E.A. Kartashova. *Constructively factoring linear partial differential operators in two variables.* J. Theoretical and Mathematical Physics, Vol. 145(2), pp.1510-1523 (2005). Springer Science+Business Media, Inc.

[2] L. Bianchi, "Lezioni di geometria differenziale", Zanichelli, Bologna, (1924)

[3] D. Grigoriev, F. Schwarz. *Factoring and Solving Linear Partial Differential Equations.* J. Computing 73 , pp.179-197 (2004)

[4] E. Kaltofen. *Factorization of polynomials.* In: Computing supplementum 4, eds: B.Buchberger, G.E.Collins and R. Loos., pp.95-113 (1982)

[5] E. Kartashova. "BK-factorization and Darboux-Laplace transformations". In: Proc. CSC'05 (The 2005 International Conference on Scientific Computing, June 20-23 Las Vegas). Ed.: H. R. Arabnia, pp. 144-150, CSREA Press, USA (2005)

[6] A.N. Leznov, M.P. Saveliev. **Group-theoretical methods for integration on non-linear dynamical systems** (Russian), Moscow, Nauka (1985). English version: Progress in Physics, 15. Birkhäuser Verlag, Basel, pp. xviii+290pp (1992)

[7] A. Shabat. "On the Laplace-Darboux theory of transformations", *Theor. Mat. Phys.*, Vol.103(1), pp.170-175 (1995)

[8] A.B. Shabat, R.I. Yamilov. "Exponential systems of type 1 and Cartan matricies." Preprint. Bashkirskii Filial Akad. Nauk SSSR, Ufa (1981)

[9] S.P.Tsarev. "An algorithm for complete enumeration of all factorizations of a linear ordinary differential operator." Proc. ISSAC'96 (1996), Y.N. Lakshman (ed.), pp.226-231 (1996)

[10] S.P.Tsarev. " Factorization on linear partial differential operators and Darboux integrability of nonlinear PDEs", Poster at ISSAC'98 (1998)

[11] G. Tzitzeica G., "Sur un theoreme de M. Darboux". Comptes Rendu de l'Academie des Aciences 150 (1910), pp.955-956; 971-974

[12] Zhiber A.V., Startsev S.Y. "Integrals, Solutions, and Existence Problems for Laplace Transformations of Linear Hyperbolic Systems." **Mathematical Notes**, Volume 74, Numbers 5-6, November 2003, pp. 803-811(9)